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Wavelet subspaces with an oversampling property

Dedicated to Jaap Korevaar on the occasion of his 70th birthday

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ABSTRACT

In the classical Shannon sampling theorem, the same sequence of functions is both orthonormal and a sampling sequence. This is not true for most wavelet subspaces in which the sampling functions and the orthonormal bases are different. However by oversampling at double the rate the property of the Shannon wavelets is extended to a much larger class which includes the Meyer wavelets. In fact together with another condition, it characterizes them.

1. INTRODUCTION

Expansions in series of orthogonal wavelets have a number of unique properties not shared by other orthogonal expansions. Two properties that make them useful are their superior pointwise convergence [10] [6] and their localization [5]. For example, the wavelet expansion of a continuous function converges uniformly; this is not true for most other systems. If the function is zero on an interval, the coefficients corresponding to that interval will be small or even zero.

However in this work we shall concentrate on another property, the sampling property of wavelet subspaces. These are similar to the classical Shannon sampling theorem [2] which recovers a band limited function from its values on the integers

$$(1.1)\qquad f(x) = \sum_{n=-\infty}^{\infty} f(n)\, S(x-n), \quad x \in \mathbb{R},$$

where $S(x) = \sin \pi x / \pi x$.

In wavelet expansions a central role is played by a multiresolution approximation, i.e. a sequence $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$ each of which is a dilation of the previous one such that $V_m \subset V_{m+1}$, $m \in \mathbb{Z}$. (More details are given in Section 2.) In [11] it was shown that under a weak hypothesis satisfied by most examples, each $f \in V_0$ can be represented by a sampling series similar to (1.1). However the series is not the wavelet expansion; rather $\{S(x-n)\}$ is a different Riesz basis of V_0. Also there are some important cases which don't work, e.g. splines of even order [3].

Recently Xia and Zhang [12] introduced a family of wavelets for which the father wavelet $\varphi(t)$ satisfies both the orthogonality condition $\int_{-\infty}^{\infty} \varphi(t)\,\varphi(t-n)\,\mathrm{d}t = \delta_{0n}$ and the sampling property $\varphi(n) = \delta_{on}$. Unfortunately, as they showed, φ cannot have compact support; and their family does not include other important families of wavelets.

In this paper we shall weaken the sampling property somewhat. Rather than look for a sampling function in V_0 we shall look for one in the dilation space V_1 and try to recover $f \in V_0$ by its values on the half integers. This gives us a sampling theorem for many additional families of wavelets composed of orthogonal sampling functions. In particular it includes the Meyer type wavelets and in fact, together with another condition, characterizes them.

This sampling property is important since it avoids integration in the approximation to $f \in L^2(\mathbb{R})$ at the finest scale

$$f_m(t) = \sum a_{mn} 2^{m/2} \varphi(2^m t - n).$$

The coefficients a_{mn} can be obtained by sampling and the others at coaser scales by the decomposition algorithm [8].

2. BACKGROUND IN WAVELETS

The theory of orthonormal wavelet bases of $L^2(\mathbb{R})$ may be found in a number of places. Detailed introductions may be found in [3] and [5] while a more complete development in $\mathbb{R}^n$ is found in [9]. Here we present a few of the basic concepts and examples most of which we shall use later.

A wavelet basis of $L^2(\mathbb{R})$ is composed of a sequence $\{\psi_{mn}\}$ of functions given by

$$\psi_{mn}(t) = 2^{m/2}\psi(2^m t - n), \quad m, n \in \mathbb{Z} \tag{2.1}$$

where ψ is a fixed function in $L^2(\mathbb{R})$, the 'mother wavelet'. Such an orthogonal basis is difficult to construct and is usually based on another function $\varphi(t)$, the 'father wavelet' or scaling function. Associated with φ is a multiresolution approximation (MRA) of $L^2(\mathbb{R})$, i.e. a nested sequence of closed subspaces $\{V_m\}_{m\in\mathbb{Z}}$ such that

$$\begin{cases} \text{(i)} & \{\varphi(t-n)\} \text{ is an orthonormal basis of } V_0, \\ \text{(ii)} & \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R}), \\ \text{(iii)} & f \in V_m \;\Leftrightarrow\; f(2\,\cdot) \in V_{m+1}, \\ \text{(iv)} & \overline{U_m V_m} = L^2(\mathbb{R}). \end{cases} \tag{2.2}$$

In addition to the condition $\varphi \in L^2(\mathbb{R})$ we shall also require $\hat{\varphi} \in L^1(\mathbb{R})$ (at least). Here $\hat{\varphi}$ denotes the Fourier transform $\hat{\varphi}(w) = \int_{-\infty}^{\infty} \varphi(t)\,\mathrm{e}^{-iwt}\,\mathrm{d}t$. Clearly $\{2^{1/2}\varphi(2t-n)\}$ must be an orthonormal basis of V_1 by (iii) and (i). Since $\varphi \in V_1$, by (ii), it must have an expansion

$$\varphi(t) = \sum_k c_k \sqrt{2}\,\varphi(2t-k), \quad c_k \in l^2. \tag{2.3}$$

This is the 'dilation equation' for φ; in terms of its Fourier transform it can be expressed as

$$\hat{\varphi}(w) = \sum_k c_k\,\mathrm{e}^{-ikw/2} 2^{-1/2} \hat{\varphi}(w/2) = m_0(w/2)\hat{\varphi}(w/2). \tag{2.4}$$

Once we have the father wavelet $\phi(t)$, we may use it to construct the 'mother wavelet' $\psi(t)$. It must be chosen such that $\{\psi(t-n)\}$ is an orthonormal basis of the space W_0, given by the orthogonal complement of V_0 in V_1. Then

$$V_1 = V_0 \oplus W_0.$$

If such a $\psi(t)$ can be found, then $2^{m/2}\psi(2^{m/2}t - n) = \psi_{nm}(t)$ is an orthonormal basis of W_n, the dilation space of W_0. Indeed from (2.2) it follows that

$$\bigoplus_{m\in Z} W_m = L^2(\mathbb{R})$$

and hence $\{\psi_{n,m}\}_{n,m\in Z}$ is an orthonormal basis of $L^2(\mathbb{R})$.

The method of finding $\psi(t)$ involving the dilation equation is straightforward; $\psi(t)$ is defined as

$$\psi(t) := \sqrt{2}\sum_k c_{1-k}(-1)^k \phi(2t-k), \tag{2.5}$$

or

$$\hat{\psi}(w) = \mathrm{e}^{-iw/2}\,\overline{m_0\left(\frac{w}{2}+\pi\right)}\,\varphi\left(\frac{w}{2}\right). \tag{2.6}$$

Then it is merely a matter of checking that the orthogonality conditions are satisfied [5].

3. MEYER TYPE WAVELETS

This example which will appear again in the next section, is an alternate way of defining the wavelets studied originally by Lemarie and Meyer [7] and subsequently by Auscher, Weiss, and Wickerhauser [1]. They have the property that the Fourier transform of the father wavelet $\varphi(t)$ has compact support.

Definition 3.1. Let the function $\varphi(t)$ be given by

$$\hat{\varphi}(w) = \left\{\int_{w-\pi}^{w+\pi} h\right\}^{1/2}$$

where h is a symmetric, positive distribution with support in $[-(\pi/3), \pi/3]$ such that $\langle h, 1\rangle = 1$. Then $\varphi(t)$ will be a father wavelet with an associated MRA $\{V_m\}$. Its associated wavelets will be denoted *Meyer type* wavelets.

We first consider some of the properties of $|\hat{\varphi}(w)|^2$. Let $[-\varepsilon, \varepsilon]$ be the smallest interval containing the support of h. It is clear that

$$\text{(3.1)}\qquad \begin{cases} \text{(i)} \quad \operatorname{supp}|\hat{\varphi}(w)|^2 = [-\pi-\varepsilon, \pi+\varepsilon] \subseteq \left[-\dfrac{4\pi}{3}, \dfrac{4\pi}{3}\right], \\ \text{(ii)} \quad |\hat{\varphi}(w)| = 1 \text{ for } |w| \leq \dfrac{2\pi}{3}, \\ \text{(iii)} \quad \sum\limits_k |\hat{\varphi}(w+2\pi k)|^2 = \int\limits_{-\infty}^{\infty} h = 1. \end{cases}$$

In order to show that φ is a father wavelet we must show the dilation equation (2.4) holds. If we define

$$m_0\left(\frac{w}{2}\right) := \sum_k \hat{\varphi}(w + 4\pi k)$$

then $m_0(w/2) = 0$ for $4\pi/3 < |w| < 8\pi/3$ and hence

$$\hat{\varphi}(w) = m_0\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right)$$

holds. Since by (3.1) (iii) and an application of Poisson's summation formula φ is orthogonal to its translates as well, it is a father wavelet. (The condition (2.2) (iv) is clear in this case.) The mother wavelet $\hat{\psi}$ (2.6) satisfies $|\hat{\psi}(w)|^2 = \int_{|w|/2-\pi}^{|w|-\pi} h$.

Each appropriate distribution (i.e. each probability measure with support in $[-\pi/3, \pi/3]$) will generate a MRA, which, as we shall see, has the oversampling property. We present a few particular cases for completeness.

Example 1. Shannon wavelet. Take $h(w) = \delta(w)$; then

$$\hat{\varphi}^2(w) = \int\limits_{w-\pi}^{w+\pi} h = \begin{cases} 1, & w-\pi \leq 0 < w+\pi \\ 0, & \text{o.w.} \end{cases}$$

This gives $\hat{\varphi}(w) = \chi_{(-\pi,\pi]}(w)$ whose inverse Fourier transform is

$$\varphi(t) = \sin \pi t/\pi t = \operatorname{sinc} t.$$

Expansions in terms of $\{\varphi(t-n)\}$ of $f \in V_0$ constitute the well-known *Shannon sampling theorem* given by (1.1).

Example 2. Original Meyer Wavelet. Lemarie and Meyer [5] defined a wavelet whose father wavelet [5, p. 137] turns out to be

$$\hat{\varphi}(w) = \begin{cases} 1, & |w| \leq \dfrac{2\pi}{3} \\ \cos\left[\dfrac{\pi}{2}\nu\left(\dfrac{3}{2\pi}|w| - 1\right)\right], & \dfrac{2\pi}{3} \leq |w| \leq \dfrac{4\pi}{3} \\ 0, & \text{o.w.} \end{cases}$$

where $\nu(x)$ is a C^k function $k \geq 1$, satisfying

$$\nu(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 1 \leq x \end{cases}$$

and $\nu(x) + \nu(1-x) = 1$.

This can be put into the form of Definition 3.1 by differention of $\hat{\varphi}(w)$.

Example 3. Exponential function. The standard example of a C^∞ function with compact support is

$$e(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1 \\ 0, & 1 \leq |x|. \end{cases}$$

Then $h(w) = C_\varepsilon\, e(w/\varepsilon)$ satisfies the required conditions for $\varepsilon \leq \pi/3$. The resulting $\hat{\varphi} \in C^\infty$ and $\varphi(t) \in S$.

Example 4. Non symmetric $h(w)$. Let $h(w) = \varepsilon^{-1}\chi_{[0,\varepsilon]}(w)$. The $\varphi(t)$ *satisfies* (2.2) *but will not be real.*

Example 5. Non positive $h(w)$. If h is taken to be

$$h(w) = \tfrac{3}{4}\delta(w+\varepsilon) - \tfrac{1}{2}\delta(w) + \tfrac{3}{4}\delta(w-\varepsilon)$$

then the conditions are satisfied except that h is not positive. This still however leads to a legitimate $\varphi(t)$.

4. A CHARACTERIZATION OF OVERSAMPLING SUBSPACES

In this section we first look for the properties of the father wavelet $\varphi(t)$ in order that $\{\varphi(2t-n)\}$ be a sampling sequence for V_0. We then show that for the Meyer family, these properties are satisfied. Finally we show that the oversampling property together with another property characterize this family.

Accordingly let $\varphi(t) = O((1+|t|)^{-1-\varepsilon})$ and $\hat{\varphi}(w) = O((1+|w|)^{-1-\varepsilon})$, $\varepsilon > 0$, and let $\{\varphi(t-n)\}$ be an orthonormal basis of V_0 such that for each $f \in V_0$,

$$f(t) = \sum_n f\left(\frac{n}{2}\right)\varphi(2t-n), \tag{4.1}$$

in the sense of $L^2(\mathbb{R})$. Then, in particular,

$$\varphi(t) = \sum_n \varphi\left(\frac{n}{2}\right)\varphi(2t-n) \tag{4.2}$$

and by taking Fourier transforms we find

$$\hat{\varphi}(w) = \begin{cases} \dfrac{1}{2}\sum_n \varphi\left(\dfrac{n}{2}\right) e^{-iwn/2}\hat{\varphi}\left(\dfrac{w}{2}\right), \\ \sum_k \hat{\varphi}(w+4\pi k)\,\hat{\varphi}\left(\dfrac{w}{2}\right). \end{cases} \tag{4.3}$$

The last equality is obtained by finding the Fourier coefficients of the 4π periodic function

$$\hat{\varphi}^*(w) := \sum_k \hat{\varphi}(w + 4\pi k),$$

which are exactly $\frac{1}{2}\varphi(n/2)$. Thus the Fourier transform of the dilation equation (2.4) is

$$\hat{\varphi}(w) = \hat{\varphi}^*(w)\,\hat{\varphi}\left(\frac{w}{2}\right). \tag{4.4}$$

This is also sufficient for (4.1) to hold.

Lemma 4.1. *Let φ be a father wavelet such that $\varphi(t) = O((1+|t|)^{-1-\varepsilon})$ and $\hat{\varphi}(w) = O((1+|w|)^{-1-\varepsilon})$ for $\varepsilon > 0$; then* (4.4) *holds for φ if and only if* (4.1) *holds for all $f \in V_0$.*

In order to show that (4.1) holds we must first show the series converges in the sense of $L^2(\mathbb{R})$. Since $\{\sqrt{2}\,\varphi(2t-n)\}$ is orthonormal, we need only show that $\{f(n/2)\} \in l^2$. Since $f \in V_0 \subseteq V_1$, it has an expansion convergent in $L^2(\mathbb{R})$ and because of the decay property of $\varphi(t)$, uniformly on bounded sets,

$$f(t) = \sum_n a_{n,1}\sqrt{2}\,\varphi(2t-n).$$

Thus we have

$$f\left(\frac{k}{2}\right) = \sum_n a_{n,1}\sqrt{2}\,\varphi(k-n)$$

and by taking the discrete Fourier transform, we find

$$\sum_k f\left(\frac{k}{2}\right) e^{iwk} = \sqrt{2}\sum_n a_{n,1}\,e^{iwn}\sum_k \varphi(k)\,e^{iwk}. \tag{4.5}$$

The right-hand side is the product of a bounded function and an $L^2(-\pi,\pi)$ function. Hence the left-hand side of (4.5) is in $L^2(-\pi,\pi)$ and $\{f(k/2)\} \in l^2$.

To show that it converges to $f(t)$ we use its expansion in V_0, which, by (4.2) is

$$\begin{aligned}
f(t) &= \sum_n a_{n,0}\,\varphi(t-n)\\
&= \sum_n a_{n,0}\sum_j \varphi\left(\frac{j}{2}\right)\varphi(2t-j-2n)\\
&= \sum_n a_{n,0}\sum_k \varphi\left(\frac{k}{2}-n\right)\varphi(2t-k)\\
&= \sum_k f\left(\frac{k}{2}\right)\varphi(2t-k).
\end{aligned}$$

The interchange of the two series is justified since the inner series is a convolution of two l^1 sequences. This is all we need since the last series converges in $L^2(\mathbb{R})$. □

We now turn to the Meyer type wavelets presented in Section 3. Their father wavelets φ were given by

$$(4.6) \qquad \hat{\varphi}(w) = \left\{ \int_{w-\pi}^{w+\pi} h(\zeta)\,\mathrm{d}\zeta \right\}^{1/2},$$

where the smallest support interval of h is $[-\varepsilon, \varepsilon]$, $0 \leq \varepsilon \leq \pi/3$.

Lemma 4.2. *Let $\hat{\varphi}(w)$ satisfy* (4.6); *then it also satisfies* (4.4).

Since $\hat{\varphi}$ is taken to be the positive square root in (4.6), we need only show that

$$(4.7) \qquad |\hat{\varphi}(w)|^2 = |\hat{\varphi}^*(w)|^2 \left|\hat{\varphi}\left(\frac{w}{2}\right)\right|^2.$$

Since $\hat{\varphi}$ has support on $[-\pi-\varepsilon, \pi+\varepsilon]$, it follows that $\hat{\varphi}^*$ has support on $\Omega = \bigcup_k [-\varepsilon + (4k-1)\,\pi, \varepsilon + (4k+1)\,\pi]$. Thus on the support of $\hat{\varphi}(w/2)$, $\hat{\varphi} = \hat{\varphi}^*$ and (4.7) becomes

$$|\hat{\varphi}(w)|^2 = |\hat{\varphi}(w)|^2 \left|\hat{\varphi}\left(\frac{w}{2}\right)\right|^2.$$

Moreover, $\hat{\varphi}(w/2) = 1$ on $[-2\pi + 2\varepsilon, 2\pi - 2\varepsilon] \supseteq [-\pi - \varepsilon, \pi + \varepsilon]$, the support of $\hat{\varphi}$. Thus (4.7) holds. □

We can also go in the opposite direction. We begin with (4.4) and try to get (4.6).

Lemma 4.3. *Let $\varphi(t)$ be a father wavelet satisfying the conditions of Lemma* 4.1 *and* (4.4); *let the support of $\hat{\varphi}$ be a bounded interval; then there is a distribution h, $\langle h, 1\rangle = 1$, with support in an interval of length $\leq 2\pi/3$ contained in $[-\pi, \pi]$ such that*

$$\int_{w-\pi}^{w+\pi} h \geq 0 \qquad \text{and} \qquad \hat{\varphi}(w) = \left[\int_{w-\pi}^{w+\pi}\right]^{1/2}.$$

We first observe that the support of $\hat{\varphi}$ must be a finite interval $[-a, b]$ where both a and b are positive. This follows from the fact that $\hat{\varphi}(0) = 1$ (this is true for all nice scaling functions, see [5]). Since $\hat{\varphi}$ is continuous, its support contains a neighborhood of the origin.

The support of $\hat{\varphi}^*$ is $\Omega = \bigcup [-a + 4\pi k, b + 4\pi k]$ and hence if $b + a \geq 4\pi$, would be all of $\mathbb{R}$. But this is impossible since by (4.4) the support of $\hat{\varphi}(w/2)$ would also have to be $[-a, b]$. We can say much more since by (4.4)

$$[-a, b] = [-2a, 2b] \cap \Omega.$$

Hence $-a+4\pi \geq 2b$ and $b-4\pi \leq -2a$ which may be expressed as

$$(4.8) \qquad a+2b \leq 4\pi, \qquad 2a+b \leq 4\pi$$

which may be added to obtain $a+b \leq 8\pi/3$.

On the other hand $2\pi \leq a+b$ since otherwise the orthogonality condition

$$(4.9) \qquad \sum_k |\hat{\varphi}(w+2\pi k)|^2 = 1, \quad w \in \mathbb{R},$$

would be violated.

Since on the support of $\hat{\varphi}$, (4.4) becomes

$$(4.10) \quad \hat{\varphi}(w) = \hat{\varphi}(w)\,\hat{\varphi}\left(\frac{w}{2}\right),$$

it follows that $\hat{\varphi}(w/2) = 1$ on $[-a, b]$ or $\hat{\varphi}(w) = 1$ on $[-(a/2), b/2]$.

We now define $h(w)$ to be

$$h(w) := \begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}w}\,|\hat{\varphi}(w+\pi)|^2, & 0 < w+\pi \\ 0, & w+\pi \leq 0 \end{cases}$$

where the derivative is in general taken in the distribution sense. It should be noticed that $h(w) = 0$ for $w < \pi - a$ or $w > b - \pi$. Furthermore, since $|\hat{\varphi}(w+\pi)|^2 + |\hat{\varphi}(w-\pi)|^2 = 1$ for $\pi - a < w < b - \pi$, it follows that

$$h(w) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}w}\,|\hat{\varphi}(w-\pi)|^2, & w-\pi < 0 \\ 0, & w-\pi \geq 0. \end{cases}$$

From these two expressions we deduce that

$$(4.11) \quad |\hat{\varphi}(w)|^2 = \int_{w-\pi}^{w+\pi} h(\zeta)\,\mathrm{d}s \geq 0, \quad w \in \mathbb{R},$$

which may be rewritten as the conclusion of the lemma, since the length of the support of h is $b+a-2\pi \leq 2\pi/3$. □

By the hypothesis $|\hat{\varphi}(w)|^2$ must be continuous but not necessarily differentiable, so that h is not necessarily a function. To be consistent with (4.10) we must take $\hat{\varphi}(w)$ to be the positive square root of $|\hat{\varphi}(w)|^2$. Its inverse Fourier transform $\varphi(t)$ is not necessarily real. However $\hat{\varphi}(w)$ is symmetric about $(-a+b)/2$ if $h(w)$ is symmetric about 0. Therefore in this case $\varphi(t)$ can be made real by shifting w by an amount $(-a+b)/2$. Then $h(w)$ satisfies the conditions of Definition 3.1, except for the positivity. We add this as a hypothesis to get

Theorem 4.1. *Let $\varphi(t)$ be a real, symmetric father wavelet such that $\varphi(t) = O(1+|t|)^{-1-\varepsilon}$, $\hat{\varphi}(w) = O(1+|w|)^{-1-\varepsilon}$, $\varepsilon > 0$ and $\hat{\varphi}$ is non increasing for $w > 0$;*

then $\varphi(t)$ is a Meyer type father wavelet if and only if (i) *$\varphi(t)$ satisfies the double sampling property* (4.2) *and* (ii) *the support of $\hat{\varphi}(w)$ is a bounded interval.*

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